

Lyapunov exponent for Lipschitz maps

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April 18, 2017

Abstract

It is well-known that the Lyapunov exponent plays a fundamental role in dynamical systems. In this note, we propose an alternative definition of Lyapunov exponent in terms of Lipschitz maps, which are not necessarily differentiable. We show that the results which are valid to standard discrete dynamical systems are also valid in this new context. Therefore, this novel approach expands the range of applications of the dynamical systems theory.

1 Introduction

Dynamical systems are extensively investigated in the literature [9, 8, 16, 5, 11, 3, 13, 12, 18, 17, 15]. Lyapunov, in his fabulous work [9], made several important contributions in the investigation of stability of motion. In fact, the Lyapunov exponent strongly characterizes the behavior of the system.

In this note, we propose an alternative definition of Lyapunov exponent in terms of Lipschitz maps. Since a Lipschitz map need not be differentiable, this novel approach increases the range of application of the dynamical systems theory.

Generalizations of Lyapunov exponents defined over continuous maps were presented in the literature [7, 2]. However, the approach of such papers is quite different from our approach. Additionally, our approach is complete in the sense that it characterizes the concepts as well as it provides natural proofs of the main results of discrete dynamical systems. Moreover, it is simple to be applied, since the hypotheses of theorems that will be presented in this paper do not require much knowledge of the map.

This note is organized as follows. Section 2 presents the main concepts that will be utilized in this work. In Section 3, we present the contributions of the paper. More precisely, we propose an alternative definition for Lyapunov exponent based on (reverse) Lipschitz maps which are not necessarily differentiable as well as we prove the results concerning discrete dynamical systems by applying this new definition. Finally, in Section 4, a briefly summary of this work is given.

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2 Preliminaries

Throughout this paper, we denote by \mathbb{R} the field of real numbers and \mathbb{R}^m is the m -dimensional vector space over \mathbb{R} . In this note we only consider discrete dynamical systems.

A function whose domain is equal to its range is called *map*. Let f be a map and x be a point of the domain of f . The *orbit* \mathcal{O}_x of x under f is the set of points $\mathcal{O}_x = \{x, f(x), f^2(x), \dots\}$, where $f^2(x) = f(f(x))$ and so on. The point x is said to be the *initial value* of the orbit. If there exists a point p in the domain of f such that $f(p) = p$ then p is called a *fixed point* of f .

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map. We say that f is *Lipschitz* if there exists a constant $c \in \mathbb{R}$, $c > 0$ (called Lipschitz constant of f), such that $\forall x, y \in \mathbb{R} \implies |f(x) - f(y)| \leq c|x - y|$, where $|\cdot|$ denotes the absolute value function on \mathbb{R} . If $\forall x, y \in \mathbb{R} \implies |f(x) - f(y)| < c|x - y|$, then f is called *strictly Lipschitz*.

Given $x \in \mathbb{R}$, the *epsilon neighborhood* $N_\epsilon(x)$ of x is defined as $N_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\}$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map and $x \in \mathbb{R}$. We say that f is *locally Lipschitz at x* if there exists an ϵ -neighborhood $N_\epsilon(x)$ of x such that f restricted to $N_\epsilon(x)$ is Lipschitz.

Here, we introduce the concept of *reverse Lipschitz map*.

Definition 2.1 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map. We say that f is *reverse Lipschitz (RL)* if there exists a constant $c \in \mathbb{R}$, $c > 0$ (called *reverse Lipschitz constant* of f) such that, $\forall x, y \in \mathbb{R} \implies |f(x) - f(y)| \geq c|x - y|$. Similarly, f is called *locally reverse Lipschitz at x* if there exists an ϵ -neighborhood $N_\epsilon(x)$ of x such that f restricted to $N_\epsilon(x)$ is reverse Lipschitz.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map and p be a fixed point of f , where p is a real number. One says that p is a *sink* (or *attracting fixed point*) if there exists an $\epsilon > 0$ such that, for all $x \in N_\epsilon(p)$, $\lim_{k \rightarrow \infty} f^k(x) = p$. On the other hand, if all points sufficiently close to p are repelled from p , then p is called a *source*. In other words, p is a source if there exists an epsilon neighborhood $N_\epsilon(p)$ such that, $\forall x \in N_\epsilon(p)$, $x \neq p$, there is $k \geq 1$ with $|f^k(x) - p| \geq \epsilon$.

3 The Results

In this section, we present the contributions of this paper. We divide the section into four subsections: stability of fixed points in \mathbb{R} , stability of periodic orbits, stability of maps on the Euclidean space \mathbb{R}^n , and a new definition of Lyapunov exponent in terms of Lipschitz maps.

3.1 Stability of fixed points in \mathbb{R}

By smooth map we consider a map which has first order derivative continuous. Let us consider the well-known result given in the following:

Theorem 3.1 [1, 10] (Stability test for fixed points) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth map. Assume that p is a fixed point of f .

- 1- If $|f'(p)| < 1$, then p is a sink.
- 2- If $|f'(p)| > 1$, then p is a source.

From here to the end of the paper, we show that exchanging the differentiability condition to the Lipschitz condition, the results which hold for standard discrete dynamical systems are also true in this new context. It is known that a Lipschitz map need not be differentiable. Thus, one can utilize this new approach for a wider class of maps.

Theorem 3.2 (Stability test for fixed points) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map and $p \in \mathbb{R}$ a fixed point of f .

- 1- If f is strictly locally Lipschitz map at p , with Lipschitz constant $c < 1$, then p is a sink.
- 2- If f is locally reverse Lipschitz map at p , with constant $r > 1$, then p is a source.

Proof: To show Item 1-), let f be a strictly locally Lipschitz map at p with Lipschitz constant $c < 1$. Then there exists an ϵ -neighborhood $N_\epsilon(p)$ of p such that $|f(x) - f(p)| < c|x - p|$ for all $x \in N_\epsilon(p)$. This fact occurs if and only if $|f(x) - f(p)|/|x - p| < c$ for all $x \in N_\epsilon(p)$ with $x \neq p$. Therefore, if $x \in N_\epsilon(p)$ then $|f(x) - f(p)| = |f(x) - p| < c|x - p| < |x - p| < \epsilon$, i.e., $f(x) \in N_\epsilon(p)$. Applying the same argument, it follows that $f^2(x), f^3(x), \dots, f^n(x), \dots$ also belong to $N_\epsilon(p)$. Next, we will prove by induction that the inequality $|f^k(x) - p| < c^k|x - p|$, $\forall x \in N_\epsilon(p)$, holds for all $k \geq 1$. It is clear that for $k = 1$ the inequality holds. Assume that the inequality is true for k : $|f^k(x) - p| < c^k|x - p|$. We must prove that $|f^{k+1}(x) - p| < c^{k+1}|x - p|$ is also true. As f is strictly locally Lipschitz at p and since $f^k(x) \in N_\epsilon(p)$ we know that $|f^{k+1}(x) - p| < c|f^k(x) - p|$. From induction hypothesis one has $|f^{k+1}(x) - p| < c^{k+1}|x - p|$ and the result follows. Since $c < 1$ it follows that $\lim_{k \rightarrow \infty} c^{k+1}|x - p| = 0$. Thus $\lim_{k \rightarrow \infty} f^k(x) = p$, i.e., p is a sink, as desired.

Let us now prove Item 2-). From hypothesis, we know that there exists an ϵ -neighborhood $N_\epsilon(p)$ of p such that $|f(x) - p| \geq r|x - p|$ for all $x \in N_\epsilon(p)$. Fix $x \in N_\epsilon(p)$, $x \neq p$. If $|f(x) - p| \geq \epsilon$, then the result follows. Otherwise, $|f(x) - p| < \epsilon$, which implies that $f(x) \in N_\epsilon(p)$. Applying again the fact that f is locally reverse Lipschitz at p , one has $|f^2(x) - p| \geq r|f(x) - p| \geq r^2|x - p|$. If $|f^2(x) - p| \geq \epsilon$, the result holds. Otherwise, $|f^2(x) - p| < \epsilon$, i.e., $f^2(x) \in N_\epsilon(p)$. Applying repeatedly this reasoning, since $r > 1$, there will be a positive integer k_0 such that $|f^{k_0}(x) - p| \geq r^{k_0}|x - p| \geq \epsilon$, and the result follows. \square

Corollary 3.3 (Stability test for fixed points) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map.

- 1- If f is strictly Lipschitz, with constant $c < 1$, then there exists only one fixed point p which is a sink.

- 2- If f is reverse Lipschitz with constant $r > 1$, then all fixed point p is a source.

Proof: Item 1-) is the well-known Banach contraction theorem on the real line (see [10, Thm. 5.2.1]). \square

3.2 Stability of periodic points in \mathbb{R}

In this subsection we deal with stability of periodic points. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map and $p \in \mathbb{R}$. Recall that p is a *periodic point* of period k (or k -periodic point) if $f^k(p) = p$, and if k is the smallest such positive integer. The orbit of p (which consists of k points) is called a *periodic orbit* of period k (or k -periodic orbit). We will denote the k -periodic orbit of p by \mathcal{O}_p^k .

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a map and p is a k -periodic point, then the orbit \mathcal{O}_p^k of p is called a *periodic sink* if p is a sink for the map f^k . Analogously, \mathcal{O}_p^k is a *periodic source* if p is a source for f^k .

Let us recall the stability criteria for periodic orbits.

Theorem 3.4 [1, 10] (*Stability test for periodic orbits*) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map. If $|f'(p_1) \cdots f'(p_k)| < 1$ then the k -periodic orbit $\mathcal{O}_p^k = \{p_1, \dots, p_k\}$ is a sink; if $|f'(p_1) \cdots f'(p_k)| > 1$ then \mathcal{O}_p^k is a source.

The following result is a new version of Theorem 3.4 based on (reverse) Lipschitz maps.

Theorem 3.5 (*Stability test for periodic orbits*) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map and $p \in \mathbb{R}$ an k -periodic point.

- 1- If f is strictly locally Lipschitz map at p , with Lipschitz constant $c < 1$, then \mathcal{O}_p^k is a periodic sink.
- 2- If f is locally reverse Lipschitz map at p , with constant $r > 1$, then \mathcal{O}_p^k is a periodic source.

Proof: 1-) Assume that f is strictly locally Lipschitz map at p . Then there exists an ϵ -neighborhood $N_\epsilon(p)$ in which f is strictly Lipschitz. Furthermore, we have shown in the proof of Theorem 3.2 that $|f^k(x) - p| < c^k|x - p|$ for all $x \in N_\epsilon(p)$. Thus, f^k is a strictly locally Lipschitz map at p with constant c^k in $N_\epsilon(p)$. Because $0 < c < 1$, it follows that $0 < c^k < 1$; therefore, from Theorem 3.2, p is a sink for the map f^k , i.e., \mathcal{O}_p^k is a periodic sink.

2-) The proof is similar to that of Item 2-) of Theorem 3.2. \square

Corollary 3.6 (*Stability test for periodic orbits*) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map.

- 1- If f is strictly Lipschitz map with constant $c < 1$, then there exists only one 1-periodic point which is a sink.

- 2- If f is reverse Lipschitz map with constant $r > 1$, then all k -periodic point is a periodic source.

Proof: Straightforward. □

3.3 Stability of fixed points in \mathbb{R}^m

As usual, we denote vectors in \mathbb{R}^m and maps on \mathbb{R}^m by boldface letters. Let us consider the m -dimensional real vector space \mathbb{R}^m endowed with a norm $\|\cdot\|$ (in particular, the Euclidean norm). Let $\mathbf{p} = (p_1, \dots, p_m), \mathbf{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$ be two points (vectors). The ϵ -neighborhood $N_\epsilon(\mathbf{p})$ of \mathbf{p} is defined by $N_\epsilon(\mathbf{p}) = \{\mathbf{v} \in \mathbb{R}^m : \|\mathbf{v} - \mathbf{p}\| < \epsilon\}$.

Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map and $\mathbf{p} \in \mathbb{R}^m$ a fixed point of \mathbf{f} , i.e., $\mathbf{f}(\mathbf{p}) = \mathbf{p}$. If there exists an ϵ -neighborhood $N_\epsilon(\mathbf{p})$ of \mathbf{p} such that $\forall \mathbf{v} \in N_\epsilon(\mathbf{p}), \lim_{k \rightarrow \infty} \mathbf{f}^k(\mathbf{v}) = \mathbf{p}$ then \mathbf{p} is called a sink (or attracting fixed point). If there exists $N_\epsilon(\mathbf{p})$ such that $\forall \mathbf{v} \in N_\epsilon(\mathbf{p})$ except for \mathbf{p} itself eventually maps outside of $N_\epsilon(\mathbf{p})$, then \mathbf{p} is called a source (or repeller).

If \mathbf{f} is a smooth map and $\mathbf{p} \in \mathbb{R}^m$, we represent \mathbf{f} in terms of its coordinates functions $\mathbf{f} = (f_1, \dots, f_m)$. Let $D\mathbf{f}(\mathbf{p})$ be the Jacobian matrix of \mathbf{f} at \mathbf{p} . With this notation in mind, we can state the following well-known result:

Theorem 3.7 (Thm. 2.11) [1] *Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map and assume that $\mathbf{p} \in \mathbb{R}^m$ is a fixed point of \mathbf{f} .*

- 1-) *If the magnitude of each eigenvalue of $D\mathbf{f}(\mathbf{p})$ is less than 1, then \mathbf{p} is a sink.*
- 2-) *If the magnitude of each eigenvalue of $D\mathbf{f}(\mathbf{p})$ is greater than 1, then \mathbf{p} is a source.*

On the vector space \mathbb{R}^m , the concept of Lipschitz map reads as follows.

Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map. We say that \mathbf{f} is Lipschitz if there exists a constant $c \in \mathbb{R}, c > 0$ such that $\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^m \implies \|\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{w})\| \leq c \|\mathbf{v} - \mathbf{w}\|$, where $\|\cdot\|$ denotes a norm over \mathbb{R}^m . If $\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^m \implies \|\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{w})\| < c \|\mathbf{v} - \mathbf{w}\|$, we say that \mathbf{f} is strictly Lipschitz.

The next result is a natural generalization of Theorem 3.2 to the Euclidean space \mathbb{R}^m .

Theorem 3.8 (Stability test for fixed points on \mathbb{R}^m) *Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map and let $\mathbf{p} \in \mathbb{R}^m$ a fixed point of \mathbf{f} .*

- 1- *If \mathbf{f} is strictly locally Lipschitz map at \mathbf{p} , with Lipschitz constant $c < 1$, then \mathbf{p} is a sink.*
- 2- *If \mathbf{f} is locally reverse Lipschitz map at \mathbf{p} , with constant $r > 1$, then \mathbf{p} is a source.*

Proof: 1-) We know that $\|\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{p})\| / \|\mathbf{v} - \mathbf{p}\| < c$ for all $\mathbf{v} \in N_\epsilon(\mathbf{p})$ with $\mathbf{v} \neq \mathbf{p}$. Therefore, if $\mathbf{v} \in N_\epsilon(\mathbf{p})$ then $\mathbf{f}(\mathbf{v}) \in N_\epsilon(\mathbf{p})$. Similarly to the

proof of Theorem 3.2, one has $\mathbf{f}^2(\mathbf{x}), \mathbf{f}^3(\mathbf{x}), \dots, \mathbf{f}^n(\mathbf{x}), \dots \in N_\epsilon(\mathbf{p})$. Furthermore, $\|\mathbf{f}^k(\mathbf{v}) - \mathbf{p}\| < c^k \|\mathbf{v} - \mathbf{p}\|$ for all $k \geq 1$. Hence, the result follows.

2-) The proof is analogous to that of Item 2-) of Theorem 3.2, so it is omitted. \square

Corollary 3.9 (*Stability test for fixed points on \mathbb{R}^m*) Let $\mathbf{f} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a map.

- 1- If \mathbf{f} is strictly Lipschitz map with Lipschitz constant $c < 1$, then all fixed point \mathbf{p} is a sink.
- 2- If \mathbf{f} is reverse Lipschitz map with constant $r > 1$, then all fixed point \mathbf{p} is a source.

Proof: Item 1-) is the well-known Banach contraction theorem (see for example [10, Thm. 5.2.1]). \square

Remark 3.10 Note that the procedure utilized in Subsection 3.2 can be easily adapted to generate analogous results for stability of periodic orbits of maps defined over \mathbb{R}^n . Since both proofs are similar, we do not present the last one here.

3.4 Lyapunov exponent

In this subsection, we define the Lyapunov number and the Lyapunov exponent in terms of Lipschitz maps. We only consider the case of maps defined over \mathbb{R} , since the procedure for maps on \mathbb{R}^n is quite similar.

We denote by $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \dots\}$ an arbitrary orbit with initial point $x_1 \in \mathbb{R}$, where $x_2 = f(x_1)$, $x_3 = f^2(x_1)$ and so on. Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth map and $x_1 \in \mathbb{R}$. Recall that the Lyapunov number $L(x_1)$ of the orbit $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \dots\}$ is defined as

$$L(x_1) = \lim_{n \rightarrow \infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n},$$

if the limit exists. The Lyapunov exponent $h(x_1)$ is defined as

$$h(x_1) = \lim_{n \rightarrow \infty} (1/n) [\ln |f'(x_1)| + \cdots + \ln |f'(x_n)|],$$

if the limit exists.

An orbit $\{x_1, x_2, \dots, x_n, \dots\}$ is called *asymptotically periodic* if it converges to a periodic orbit when $n \longrightarrow \infty$. In other words, there exists a periodic orbit $\{y_1, y_2, \dots, y_k, y_1, y_2, \dots, y_k, \dots\}$ such that $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$.

Let us recall the result due to Rademacher:

Theorem 3.11 (*Rademacher's theorem*) [Thm. 3.1.6.] [4] (see also [14]) If $\mathbf{f} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a Lipschitz map, then \mathbf{f} is differentiable at Lebesgue almost all points of \mathbb{R}^m .

A variant of this result is given below.

Theorem 3.12 (Rademacher's theorem)[6, Thm. 3.1] *Let $\Omega \subset \mathbb{R}^m$ be an open set, and let $\mathbf{f}: \Omega \rightarrow \mathbb{R}^n$ be Lipschitz. Then \mathbf{f} is differentiable at almost every point (Lebesgue) in Ω .*

From Rademacher's theorem, we can guarantee that a Lipschitz map $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at a set $X = \mathbb{R} - Y$, where the set Y has zero Lebesgue measure. In this context, we can define the Lyapunov number (and exponent) as well as the concept of asymptotically periodic orbit based on Lipschitz maps.

Definition 3.1 *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz map and assume that $\mathcal{O}_{x_1} \subset X$. Then the Lyapunov number $L(x_1)$ of the orbit $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \dots\}$ is defined as*

$$L(x_1) = \lim_{n \rightarrow \infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n},$$

if the limit exists. The Lyapunov exponent $h(x_1)$ is defined as

$$h(x_1) = \lim_{n \rightarrow \infty} (1/n) [\ln |f'(x_1)| + \cdots + \ln |f'(x_n)|],$$

if the limit exists.

Recall that a map $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be *locally Lipschitz* on an open interval $(a, b) \subset \mathbb{R}$ if f restricted to (a, b) is Lipschitz. In terms of locally Lipschitz maps, we have the following variant to Definition 3.1.

Definition 3.2 *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a locally Lipschitz map in an (nondegenerate) open interval (a, b) , and assume that $\mathcal{O}_{x_1} \subset (a, b) \cap X$. Then the Lyapunov number $L(x_1)$ of the orbit $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \dots\}$ is defined as*

$$L(x_1) = \lim_{n \rightarrow \infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n},$$

if the limit exists. The Lyapunov exponent $h(x_1)$ is defined as

$$h(x_1) = \lim_{n \rightarrow \infty} (1/n) [\ln |f'(x_1)| + \cdots + \ln |f'(x_n)|],$$

if the limit exists.

Let us reformulate the concept of *asymptotically periodic orbit* in terms of Lipschitz maps.

Definition 3.3 *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz map. An orbit $\{x_1, x_2, \dots, x_n, \dots\}$ is called *asymptotically periodic* if it converges to a periodic orbit when $n \rightarrow \infty$. In other words, there exists a periodic orbit $\{y_1, y_2, \dots, y_k, y_1, y_2, \dots, y_k, \dots\}$ such that $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$.*

The following result is a variant of [1, Theorem 3.4] based on Lipschitz maps.

Theorem 3.13 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz map with first derivative continuous in the set X . Assume that the orbit $\mathcal{O}_{x_1} = \{x_1, x_2, \dots, x_n, \dots\} \subset X$ satisfies $f'(x_i) \neq 0$ for all $i = 1, 2, \dots$. If \mathcal{O}_{x_1} is asymptotically periodic to the periodic orbit $\mathcal{O}_{y_1} = \{y_1, y_2, \dots, y_k, y_1, y_2, \dots, y_k, \dots\}$, then $h(x_1) = h(y_1)$ if both Lyapunov exponent exist.

Proof: Since f is a Lipschitz map, applying Rademacher's theorem with $n = m = 1$, it follows that f has derivative in the set X . As $\mathcal{O}_{x_1} \subset X$, we can guarantee the derivative of all point of \mathcal{O}_{x_1} . From here, the proof is similar to the proof of [1, Theorem 3.4]. We present it here for completeness.

Assume that $k = 1$; then $\lim_{n \rightarrow \infty} x_n = y_1$. Since the derivative is continuous it follows that $\lim_{n \rightarrow \infty} f'(x_n) = f'(y_1)$. Moreover, one has $\lim_{n \rightarrow \infty} \ln |f'(x_n)| = \ln |f'(y_1)|$. Therefore, $h(x_1) = \lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n \ln |f'(x_i)| = \ln |f'(y_1)| = h(x_1)$. If $k > 1$, we know that y_1 is a fixed point of f^k and \mathcal{O}_{x_1} is asymptotically periodic under f^k to \mathcal{O}_{y_1} . Applying the reasoning above to x_1 and f^k it follows that $h(x_1) = \ln |(f^k)'(y_1)|$. It is known that if L is the Lyapunov number of \mathcal{O}_{x_1} under the map f , then the Lyapunov number of \mathcal{O}_{x_1} under the map f^k is L^k (see [1, Ex T3.1]). Then the Lyapunov exponent $h(x_1)$ of x_1 under f equals $h(x_1) = 1/k \ln |(f^k)'(y_1)| = h(y_1)$. The proof is complete. \square

A version of Theorem 3.13 to locally Lipschitz maps is given below.

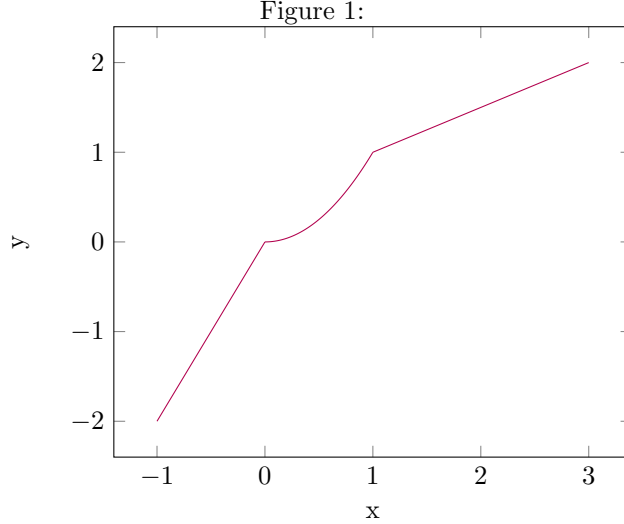
Theorem 3.14 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a locally Lipschitz map on the open interval (a, b) with first derivative continuous in (a, b) . Assume that the orbit $\mathcal{O}_{x_1} \subset (a, b) \cap X$ satisfies $f'(x_i) \neq 0$ for all $i = 1, 2, \dots$. If \mathcal{O}_{x_1} is asymptotically periodic to the periodic orbit $\mathcal{O}_{y_1} = \{y_1, y_2, \dots, y_k, y_1, y_2, \dots, y_k, \dots\}$, then $h(x_1) = h(y_1)$ if both Lyapunov exponent exist.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map, and let \mathcal{O}_{x_1} be a bounded orbit of f . Recall that the orbit is *chaotic* if \mathcal{O}_{x_1} is not asymptotically periodic and if the Lyapunov exponent $h(x_1)$ is greater than zero. In terms of Lipschitz maps one has the following new definition for chaotic orbits:

Definition 3.4 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz map with first derivative continuous at X , and let \mathcal{O}_{x_1} be a bounded orbit of f . We call the orbit *chaotic* if

1. \mathcal{O}_{x_1} is not asymptotically periodic;
2. the Lyapunov exponent $h(x_1)$ is greater than zero.

To finish this section, we give two examples of maps which are locally Lipschitz and Lipschitz, shown in Examples 3.1 and 3.2 respectively, but they are not differentiable.



Example 3.1 Let us consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$x_{n+1} = f(x_n) = \begin{cases} 2x_n, & x_n < 0 \\ x_n^2, & 0 \leq x_n < 1 \\ 0.5x_n + 0.5, & x_n \geq 1 \end{cases} ,$$

Figure 1 shows the graphic of the map f . Note that f is not differentiable at $p = 0$ and $p = 1$, but it is locally Lipschitz for the open intervals $(-\infty, 0)$, $(0, 1)$ $(1, +\infty)$, so our method can be applied.

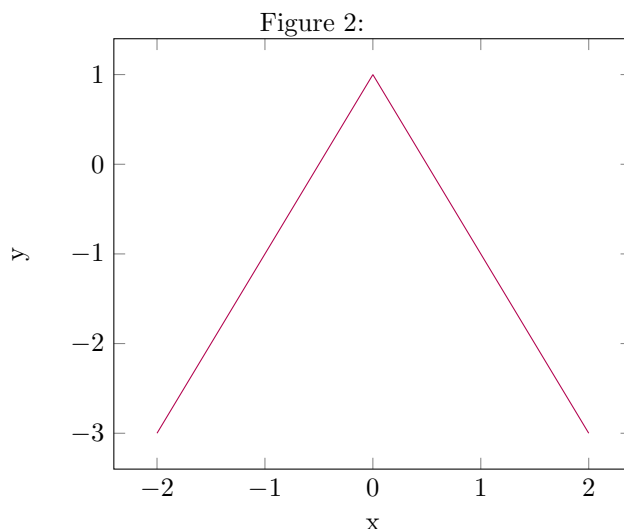
Example 3.2 Here, let us consider the family of Lipschitz maps $g_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$x_{n+1} = g_{a,b}(x_n) = a|x_n| + b,$$

where a, b are real numbers. For $a = -2$ and $b = 1$, the graphic of $g_{-2,1}$ is shown in Figure 2. Note that the map $g_{-2,1}$ is Lipschitz, so our method can be applied, but it is not differentiable.

4 Final Remarks

We have proposed an alternative definition for the Lyapunov exponent in terms of Lipschitz maps, which are not necessarily differentiable. Furthermore, we have shown that the results which are valid to standard discrete dynamical systems are also true in this new context. This novel approach expands the range of application of the dynamical systems theory and it is simple to be applied.



Acknowledgements

This research has been partially supported by the Brazilian Agencies CAPES and CNPq.

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